CHAPTER 5

5.1 The Natural Logarithmic Function: Differentiation

NATURAL LOGARITHMIC FUNCTION

the natural logarithmic function is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt \quad , \quad x>0$$

domain = all positive real numbers.

think of natural logarithmic function as an antiderivative given by the differential equation

$$\frac{dy}{dx} = \frac{1}{x}$$

PROPERTIES OF THE NATURAL LOGARITHMIC FUNCTION

The natural logarithmic function has the following properties

1) The domain is $(0,\infty)$, range is $(-\infty,\infty)$
2) ~~the~~ the function is continuous, increasing
3) The graph is concave downward

LOGARITHMIC PROPERTIES

If a and b are positive numbers and n is rational, then the following properties are true.

1) $\ln(1) = 0$
2) $\ln(ab) = \ln a + \ln b$
3) $\ln(a^n) = n \ln a$
4 $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

a) $\ln \frac{10}{9} = \ln 10 - \ln 9$

b) $\ln \sqrt{3x+2} = \frac{1}{2} \ln(3x+2)$

c) $\ln \frac{6x}{5} = \ln(6x) - \ln 5$

$= \ln 6 + \ln x - \ln 5$

d) $\ln \frac{(x^2+3)^2}{x \sqrt[3]{x^2+1}} = \ln (x^2+3)^2 - \ln(x \sqrt[3]{x^2+1})$

$= 2 \ln(x^2+3) - \ln x - \frac{1}{2} \ln(x^2+1)$

DEFINITION OF e

$\ln e \int_1^e \frac{1}{t}\, dt = 1$

x	$\frac{1}{e^3} \approx 0.050$	$e^0 = 1$
$\ln x$	-3	0

logorthms' x values are integer powers of e

$\ln|t| \Big]_1^e$

1

$\cancel{\ln e}(\ln e - \ln 1)$

$1 - 0 = 1$

The DERIVATIVE OF THE NATURAL LOGARITHMIC FUNCTION

Let u be a differentiable function of x

1) $\frac{d}{dx}[\ln x] = \frac{1}{x}, \; x > 0$

2) $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u} \quad u > 0$

Examples

a. $\frac{d}{dx}[\ln(2x)] = \frac{u'}{u} = \frac{2}{2x} = \frac{1}{x}$ → $u = 2x$

b. $\frac{d}{dx}[\ln(x^2+1)] = \frac{u'}{u} = \frac{2x}{x^2+1}$ → $u = x^2+1$

c. $\frac{d}{dx}[x \ln x] = x\left(\frac{d}{dx}[\ln x]\right) + (\ln x)\left(\frac{d}{dx}[x]\right) = x\left(\frac{1}{x}\right) + (\ln x) = \boxed{1 + \ln x}$ → product rule

d. $\frac{d}{dx}[(\ln x)^3] = 3(\ln x)^2 \frac{d}{dx}[\ln x]$ → chain rule

$= 3(\ln x)^2 \frac{1}{x}$

Example 4: log. Properties as Aids to Differentiation

Differentiate $f(x) = \ln\sqrt{x+1}$

$$f(x) = \ln\sqrt{x+1} = \ln (x+1)^{1/2} = \frac{1}{2}\ln(x+1)$$

$$f'(x) = \frac{1}{2}\left(\frac{1}{x+1}\right) = \frac{1}{2(x+1)}$$

Example 5

Differentiate $f(x) = \ln \frac{x(x^2+1)^2}{\sqrt{2x^3-1}}$

$$\cancel{f(x)} = \ln x + 2\ln(x^2+1) - \frac{1}{2}\ln(2x^3-1)$$

$$f'(x) = \frac{1}{x} + 2\left(\frac{2x}{x^2+1}\right) - \frac{1}{2}\left(\frac{6x^2}{2x^3-1}\right)$$

$$= \frac{1}{x} + \frac{4x}{x^2+1} - \frac{3x^2}{2x^3-1}$$

Example 6

Find the derivative of $y = \frac{(x-2)^2}{\sqrt{x^2+1}}$ $x \neq 2$

$$\ln y = \ln \frac{(x-2)^2}{\sqrt{x^2+1}} = 2\ln(x-2) - \frac{1}{2}\ln(x^2+1)$$

$$\frac{y'}{y} = 2\left(\frac{1}{x-2}\right) - \frac{1}{2}\left(\frac{2x}{x^2+1}\right)$$

$$y' = \left[\frac{x^2+2x+2}{(x-2)(x^2+1)}\right] y = \boxed{\frac{(x-2)(x^2+2x+2)}{(x^2+1)^{3/2}}}$$

DERIVATIVE INVOLVING ABSOLUTE VALUE

$$\boxed{\frac{d}{dx}\left[\ln |u|\right] = \frac{u'}{u}}$$

e.g.

$f(x) = \ln|\cos x|$

$$\frac{d}{dx}\left[\ln|\cos x|\right] = \frac{u'}{u}$$

$$= \frac{-\sin x}{\cos x}$$

$$= -\tan x$$

5.2 THE NATURAL LOG FUNCTION: INTEGRAT

LOG RULE FOR INTEGRATION

let u be a differentiable function of x

1) $\int \frac{1}{x}\,dx = \ln|x| + C$ 2) $\int \frac{1}{u}\,du = \ln|u| + C$

alternative form of log Rule: $\int \frac{u'}{u}\,dx = \ln|u| + C$

because $du = u'dx$

Ex. 1

$$\int \frac{2}{x}\,dx = 2\int \frac{1}{x}\,dx$$
$$= 2\ln|x| + C$$
$$= \ln(x^2) + C$$

(because x^2 can't be negative)

Ex. 2

$$\int \frac{1}{4x-1}\,dx$$

$u = 4x-1$

$du = 4\,dx$

$$\int \frac{1}{4x-1}\,dx = \frac{1}{4}\int \frac{1}{4x-1}\,4\,dx$$
$$= \frac{1}{4}\int \frac{1}{u}\,du$$
$$= \frac{1}{4}\ln|u| + C$$
$$= \frac{1}{4}\ln|4x-1| + C$$

Ex. 3 FINDING AREA WITH LOG RULE

Find area of region bounded by graph $y = \frac{x}{x^2+1}$, x axis, and line $x=3$

$$\int_0^3 \frac{x}{x^2+1}\,dx \qquad u = x^2+1 \quad du = 2x\,dx$$

$$\int_0^3 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_0^3 \frac{2x}{x^2+1}\,dx$$

$$= \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\left[\ln(x^2+1)\right]_0^3$$

$$= \frac{1}{2}(\ln 10 - \ln 1)$$

$$= \frac{1}{2}(\ln 10) \approx 1.151$$

Ex. 4 RECOGNIZING QUOTIENT FORMS OF THE LOG RULE

a. $\int \frac{3x^2+1}{x^3+x}\,dx = \ln|x^3+x| + C$

b. $\int \frac{\sec^2 x}{\tan x}\,dx = \ln|\tan x| + C$

c. $\int \frac{x+1}{x^2+2x}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x}\,dx = \frac{1}{2}\ln|x^2+2x| + C$

d. $\int \frac{1}{3x+2}\,dx = \frac{1}{3}\int \frac{3}{3x+2}\,dx = \frac{1}{3}\ln|3x+2| + C$

Ex. 5 USING LONG DIVISION BEFORE INTEGRATING

Find $\int \frac{x^2+x+1}{x^2+1}\,dx$

$$\begin{array}{r|l} & 1 \\ x^2+1 & \overline{x^2+x+1} \\ & \underline{x^2+1} \\ & x \end{array} = 1+\frac{x}{x^2+1}$$

$$\int \frac{x^2+x+1}{x^2+1}\,dx = \int \left(1+\frac{x}{x^2+1}\right)dx$$

$$= \int 1\,dx + \frac{1}{2}\int \frac{2x}{x^2+1}\,dx$$

$$= x + \frac{1}{2}\ln(x^2+1) + C$$

$x = (u+1)^3$

$dx = 3(u+1)^2\, du$

Ex. 6 CHANGE OF VARIABLES WITH THE LOG RULE

Find $\int \frac{2x}{(x+1)^2}\, dx$

$u = x+1 \qquad du = dx \qquad x = u-1$

$$\int \frac{2x}{(x+1)^2}\, dx = \int \frac{2(u-1)}{u^2}\, du$$

$$= 2\int \left(\frac{u}{u^2} - \frac{1}{u^2} \right) du$$

$$= 2\int \frac{du}{u} - 2\int u^{-2}\, du$$

$$= 2 \ln|u| - 2\left(\frac{u^{-1}}{-1}\right) + C$$

$$= 2\ln|u| + \frac{2}{u} + C$$

$$= 2\ln|x+1| + \frac{2}{x+1} + C$$

GUIDELINES FOR INTEGRATION

1. Learn basic list of integration formulas
   We have 12: Power Rule, Log Rule, 10 trig rules.

2. Find an integration formula that resembles all or part of the integrand, and, by trial and error, find a choice of u that will make the integrand conform to the formula

3. If you cannot find a u-substitution that works, try altering the integrand.
   You can try a trig identity, multiplication and division, addition and subtraction, or long division

Ex. 7.

$$\frac{dy}{dx} = \frac{1}{x \ln x}$$

$$y = \int \frac{1}{x \ln x} dx = \int \frac{1/x}{\ln x} dx$$

$$= \int \frac{u'}{u} dx$$

$$= \ln u + c$$

$$= \ln|\ln x| + c$$

INTEGRALS OF TRIGONOMETRIC FUNCTIONS

Ex. 8

Find $\int \tan x \, dx$

$$= \int \frac{\sin x}{\cos x} \, dx \qquad u = \cos x$$

$$u' = -\sin x$$

$$\int \tan x \, dx = -\int \frac{-\sin x}{\cos x} \, dx$$

$$= -\int \frac{u'}{u} \, dx$$

$$= \ln|u| + C$$

$$= -\ln|\cos x| + C$$

Ex. 9

$$\int \sec \, dx = \int \sec x \times \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$u = \sec x + \tan x$

$u' = \sec x \tan x + \sec^2 x$

$$= \int \frac{u'}{u} \, dx$$

$$= \ln|\sec x + \tan x| + C$$

INTEGRALS OF SIX BASIC TRIG FUNCTIONS

$\int \sin u \, du = -\cos u + C$ $\qquad$ $\int \cos u \, du = \sin u + C$

☆ $\int \tan u \, du = -\ln|\cos u| + C$ $\qquad$ $\int \cot u \, du = \ln|\sin u| + C$

$\int \sec u \, du = \ln|\sec u + \tan u| + C$ $\qquad$ $\int \csc u \, du = -\ln|\csc u + \cot u| + C$

Ex. 10 Integrating Trig Functions

$$\int_0^{\pi/4} \sqrt{1+\tan^2 x} \, dx \qquad 1+\tan^2 x = \sec^2 x$$

$$= \int_0^{\pi/4} \sqrt{\sec^2 x} \, dx$$

$$= \int_0^{\pi/4} \sec x \, dx$$

$$= \ln|\sec x + \tan x| \Big]_0^{\pi/4}$$

$$= \ln(\sqrt{2}+1) - \ln 1$$

$$\approx 0.881$$

EX. 11
FINDING AVERAGE value

find average value of $f(x) = \tan x$ on $[0, \frac{\pi}{4}]$

$$A.V = \frac{1}{\pi/4 - 0} \int_0^{\pi/4} \tan x \, dx$$

$$= \frac{4}{\pi} \int_0^{\pi/4} \tan x \, dx$$

$$= \frac{4}{\pi} \left[-\ln|\cos x| \right]_0^{\pi/4}$$

$$= -\frac{4}{\pi} \left[\ln\left(\frac{\sqrt{2}}{2}\right) - \ln(1) \right]$$

$$= -\frac{4}{\pi} \ln\left(\frac{\sqrt{2}}{2}\right)$$

$$\approx 0.441$$

5.3 INVERSE FUNCTIONS

DEFINITION OF INVERSE FUNCTION

A function g is the inverse function of f if
$f(g(x)) = x$ and
$g(f(x)) = x$

g is denoted f^{-1}

Verifying inverse functions:

$f(x) = 2x^3 - 1$ $\quad$ $g(x) = \sqrt[3]{\frac{x+1}{2}}$

$$f(g(x)) = 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 = x$$

$$g(f(x)) = \sqrt[3]{\frac{(2x^3-1)+1}{2}} = x$$

REFLECTIVE PROPERTY OF INVERSE FUNCTIONS

The graph of f contains point (a,b) if and only if the graph of f^{-1} contains point (b,a)

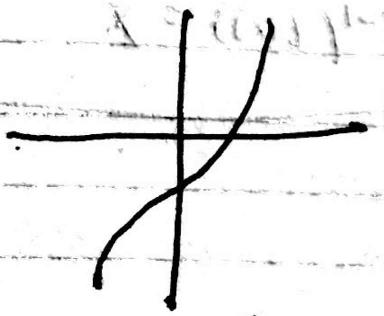
$$f^{-1}(b) = f^{-1}(f(a)) = a$$

Existence of an inverse Function

1. A function has an inverse function if and only if it is one to one
2. If f is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function

e.g. Does it have an inverse function?

$f(x) = x^3 + x - 1$

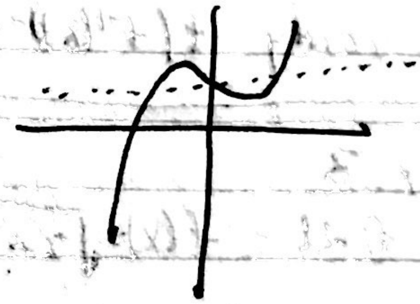


(increasing on entire domain)

It is strictly monotonic and thus has an inverse function

Yes

$f(x) = x^3 - x + 1$



Doesn't pass horizontal line test. It is not 1-to-1

$f(-1) = f(1) = f(0) = 1$

No

GUIDELINES FOR FINDING AN INVERSE FUNCTION

1. Determine whether $y=f(x)$ has an inverse function
2. Solve for x as a function of y
   $x=g(y)=f^{-1}(y)$
3. Interchange x and y. The resulting equation is $y=f^{-1}(x)$
4. Define the Domain of f^{-1} as the range of f
5. Verify $f(f^{-1}(x))=x$ and $f^{-1}(f(x))=x$

ex. 3

find $f(x)=\sqrt{2x-3}$

$f'(x)=\frac{1}{\sqrt{2x-3}}$ monotonic ⇒ positive on entire domain ⇒ has inverse function

$\sqrt{2x-3}=y$

$x=\frac{y^2+3}{2}$ solve for x

$y=\frac{x^2+3}{2}$ switch x and y

$\boxed{f^{-1}(x)=\frac{x^2+3}{2}}$

ex. 4 Testing whether function is one to one

$f(x) = \sin x$

⇒ show function is not one-to-one on entire real line.

⇒ Show $[-\pi/2, \pi/2]$ is largest interval where f is strictly monotonic

Clear f is not one-to-one

$\sin(0) = 0 = \sin(\pi)$

f is increasing on open interval $(-\pi/2, \pi/2)$

$f'(x) = \cos x$

is positive $(-\pi/2, \pi/2)$

DERIVATIVE OF AN INVERSE FUNCTION

CONTINUITY AND DIFFERENTIABILITY OF INVERSE FUNCTIONS

$f \Rightarrow$ function whose domain is an interval I
If f has an inverse function,
following is true

1) If f is continuous on its domain, then f^{-1} is continuous on its domain
2) If f is increasing on its domain, then f^{-1} is increasing on its domain
3) If f is decreasing on its domain, then f^{-1} is decreasing on its domain
4) If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$

THE DERIVATIVE OF AN INVERSE FUNCTION

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0$$

EX. 5

$$f(x) = \frac{1}{4}x^3 + x - 1$$

a) what is the value of $f^{-1}(x)$ when $x=3$

b) what is the value of $(f^{-1})'(x)$ when $x=3$

a) $f(x)=3$ when $x=2$ $\Rightarrow$

$f^{-1}(3)=2$

b) $$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(2)} = \frac{1}{\frac{3}{4}(2^2)+1} = \frac{1}{4}$$

If $y = g(x) = f^{-1}(x)$

$f(y) = x$

$f'(y) = \frac{dx}{dy}$

$$g'(x) = \frac{dy}{dx} = \frac{1}{f'(g(x))} = \frac{1}{f'(y)} = \frac{1}{(dx/dy)}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{dx/dy}}$$

Ex. 6 GRAPHS OF INVERSE FUNCTIONS HAVE RECIPROCAL SLOPES

$f(x) = x^2 \quad x \geq 0$

$f^{-1}(x) = \sqrt{x}$

Show slopes at graphs at f and f^{-1} are reciprocal at following points

a. (2,4) and (4,2)

b. (3,9) and (9,3)

a. $f'(2) = 2(2) = 4$ At (2,4)

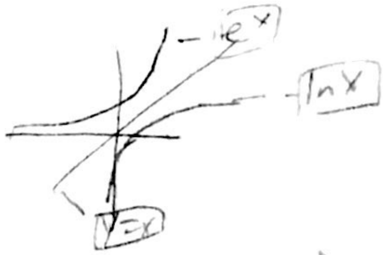
$(f^{-1})'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2(2)} = \frac{1}{4}$ At (4,2)

b. At (3,9),

$f'(3) = 2(3) = 6$

At (9,3)

$(f^{-1})'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}$



5.4 EXPONENTIAL FUNCTIONS: DIFFERENTIATION AND INTEGRATION

The Natural Exponential Function

$f(x) = \ln x$ ∃ natural exponential function denoted by

$f^{-1}(x) = e^x$

$y = e^x$ if and only if $x = \ln y$

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln x} = x$$

Example 1

solve $7 = e^{x+1}$

$7 = e^{x+1}$

$\ln 7 = \ln(e^{x+1})$

$\ln 7 = x + 1$

$-1 + \ln = x$

$x \approx 0.946$

Example 2

solve $\ln(2x-3) = 5$

$\ln(2x-3) = 5$

$e^{\ln(2x-3)} = e^5$

$2x - 3 = e^5$

$x = \frac{1}{2}(e^5 + 3)$

$x \approx 75.707$

OPERATIONS WITH EXPONENTIAL FUNCTIONS

1. $e^a e^b = e^{a+b}$
2. $\frac{e^a}{e^b} = e^{a-b}$

PROPERTIES OF THE NATURAL EXPONENTIAL FUNCTIONS

1. The domain of $f(x) = e^x$ is $(-\infty, \infty)$ and range is $(0, \infty)$
2. The function $f(x) = e^x$ is continuous, increasing, and one to one on its entire domain
3. The graph of $f(x) = e^x$ is concave upward on its entire domain
4. $\lim_{x \to -\infty} e^x = 0$ and $\lim_{x \to \infty} e^x = \infty$

Derivatives of Exponential Functions

DERIVATIVES OF THE NATURAL EXPONENTIAL FUNCTION

1. $\frac{d}{dx}[e^x] = e^x$
2. $\frac{d}{dx}[e^u] = e^u \frac{du}{dx}$

$\Rightarrow$ u is differentiable function of x

e.g. 3

a) $\frac{d}{dx}[e^{2x-1}] = e^u \frac{du}{dx} = 2e^{2x-1}$ $\quad u = 2x-1$

b) $\frac{d}{dx}[e^{-3/x}] = e^u \frac{du}{dx} = \frac{3}{x^2} e^{-3/x} = \frac{3e^{-3/x}}{x^2}$

$u = \frac{3}{x}$

e.g. 4

$f(x) = xe^x$

$$f'(x) = x(e^x) + e^x(1)$$
$$= e^x(x+1)$$

INTEGRATION RULES FOR EXPONENTIAL FUNCTIONS	
1. $\int e^x dx = e^x + C$	2. $\int e^u du = e^u + C$

e.g. 7 Find $\int e^{3x+1} dx$ $\Rightarrow$ $u = 3x+1$, $du = 3dx$

$$\int e^{3x+1} dx = \frac{1}{3}\int e^{3x+1}(3)dx$$
$$= \frac{1}{3}\int e^u du$$
$$= \frac{1}{3}e^u + C$$
$$= \frac{e^{3x+1}}{3} + C$$

e.g. 8 $\int 5xe^{-x^2} dx$

$u = -x^2$

$du = -2x\, dx$

$x\, dx = -\frac{du}{2}$

$$\int 5xe^{-x^2} dx = \int 5e^{-(x^2)}(x dx)$$
$$= \int 5e^u\left(-\frac{du}{2}\right)$$
$$= -\frac{5}{2}\int e^u du = -\frac{5}{2}e^u + C = -\frac{5}{2}e^{(-x^2)} + C$$

$e^u \quad u=\frac{1}{x} \quad du=-\frac{1}{x^2}$

a. $\int \frac{e^{1/x}}{x^2}\,dx = -\int \underbrace{e^{1/x}}_{e^u} \underbrace{\left(-\frac{1}{x^2}\right) dx}_{du}$

$= -e^{1/x} + C$

b. $\int \sin x \, e^{\cos x}\,dx = -\int e^{\cos x}(-\sin x\,dx)$

$= -e^{\cos x} + C$

$u = \cos x$
$du = -\sin dx$

EX. 10 FINDING AREAS BOUNDED BY EXPONENTIAL FUNCTION

(a) $\int_0^1 e^{-x}\,dx$

$= -e^{-x}\Big]_0^1$

$= -e^{-1} - (-1)$

$= 1 - \frac{1}{e}$

≈ 0.632

(b) $\int_0^1 \frac{e^x}{1+e^x}\,dx = \ln(1+e^x)\Big]_0^1$

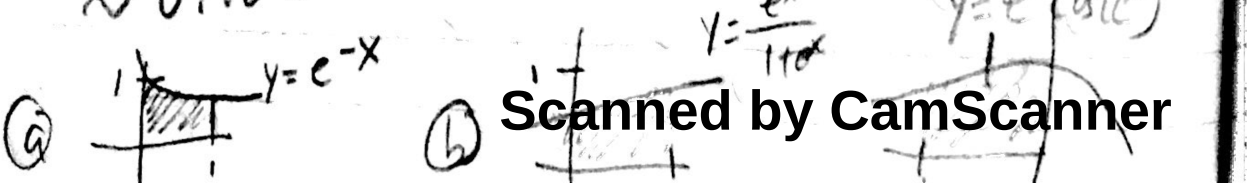
$= \ln(1+e) - \ln 2$

≈ 0.620

(c) $\int_{-1}^{0} [e^x \cos(e^x)]\,dx = \sin(e^x)\Big]_{-1}^{0}$

$= \sin 1 - \sin(e^{-1})$

≈ 0.482



5.5 BASES OTHER THAN E AND APPLICATIONS

Bases other than e

DEFINITION OF EXPONENTIAL FUNCTION TO BASE a

If a is positive real number ($a \neq 1$) and x is any real number, exponential function to the base a is denoted a^x,

$$a^x = e^{(\ln a)x}$$

$a = 1 \qquad y = 1^x = 1$

1. $a^0 = 1$
2. $a^x a^y = a^{x+y}$
3. $\frac{a^x}{a^y} = a^{x-y}$
4. $(a^x)^y = a^{xy}$

Eg. 1 Radioactive Half-life Model

The half-life of Carbon-14 is about 5715 yrs.

A sample contains 1 gram of Carbon-14.

How much will be present in 10,000 years?

e.g. $y = \left(\frac{1}{2}\right)^{5715/5715} = \frac{1}{2}$ gram

$y = \left(\frac{1}{2}\right)^{10,000/5715} \approx 0.3$ gram

DEFINITION OF LOGARITHMIC FUNCTON TO BASE a

If a is a positive real number ($a \neq 1$) and x is any positive real number, then the logarithmic function to the base a, denoted by $\log_a x$ defined as

$$\log_a x = \frac{1}{\ln a} \ln x = \frac{\ln x}{\ln a}$$

1. $\log_a 1 = 0$
2. $\log_a xy = \log_a x + \log_a y$
3. $\log_a x^n = n \log_a x$
4. $\log_a \frac{x}{y} = \log_a x - \log_a y$

PROPERTIES OF INVERSE FUNCTIONS

1. $y = a^x$ if/only if $x = \log_a y$
2. $a^{\log_a x} = x$ for $x > 0$
3. $\log_a a^x = x$ for all x

Bases Other Than e

a. $3^x = \frac{1}{81}$

$\log_3 3^x = \log_3 \frac{1}{81}$

$x = \log_3 3^{-4}$

$\boxed{x = -4}$

b. $\log_2 x = -4$

$\log_2 x = -4$

$2^{\log_2 x} = 2^{-4}$

$x = \frac{1}{2}$

DIFFERENTIATION AND INTEGRATION

$\boxed{\text{Derivatives for Bases Other Than } e}$

1. $\frac{d}{dx}[a^x] = (\ln a)a^x$
2. $\frac{d}{dx}[a^u] = (\ln a) a^u \frac{du}{dx}$
3. $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$
4. $\frac{d}{dx}[\log_a u] = \frac{1}{(\ln a)u} \frac{du}{dx}$

E.s. 3 differentiating functions to other bases

a. $y = 2^x$

$y' = \frac{d}{dx}[2^x] = (\ln 2)2^x$

b. $y = 2^{3x}$

$y' = \frac{d}{dx}[2^{3x}] =$
$(\ln 2)\ 2^{3x}(3)$
$= (3\ln 2)2^{3x}$

c. $y = \log_{10} \cos x$

$y' = \frac{d}{dx}[\log_{10} \cos x]$

$= \frac{-\sin x}{\ln 10 \cos x}$

$= -\frac{1}{\ln 10} \tan x$

$$\boxed{\int a^x\, dx = \left(\frac{a^x}{\ln a}\right) + C}$$

Find $\int 2^x dx$

$\int 2^x dx = \frac{1}{\ln 2} 2^x + C$

THE POWER RULE FOR REAL EXPONENTS

1. $\frac{d}{dx}[x^n] = nx^{n-1}$

2. $\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$

E.g. 5 Comparing Variables & Constants

a. $\frac{d}{dx}[e^e] = 0$ Constant rule

b. $\frac{d}{dx}[e^x] = e^x$ exponential rule

c. $\frac{d}{dx}[x^e] = ex^{e-1}$ Power rule

d. $y = x^x$

$\ln y = \ln x^x$

$\ln y = x \ln x$

$\frac{y'}{y} = x\frac{1}{x} + (\ln x)(1) = 1 + \ln x$

$y' = y(1+\ln x) = x^x(1+\ln x)$

Applications of Exponential Functions

$$A = P\left(1+\frac{r}{n}\right)^{n}$$

Limit involving e

$$\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^{x} = e}$$

$$A = \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{n}$$

$$= P\lim_{n\to\infty}\left[\left(1+\frac{1}{n/r}\right)^{n/r}\right]^{r}$$

$$= P\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{r}$$

$$= Pe^{r}$$

SUMMARY OF COMPOUND INTEREST FORMULAS

P = amount of deposit

t = number of years

n = number of compoundings per year

A = balance after t years

r = annual interest rate (decimal form)

1. Compounded n times per year: $A = P\left(1+\frac{r}{n}\right)^{nt}$

2. Compounded continuously: $A = Pe^{rt}$

example:

A deposit of $2500 is made in an account that pays an annual interest rate of 5%. Find the balance in the amount at the end of 5 years if interest is compounded

a) quarterly b) monthly c) Continuously

a)

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$= 2500\left(1+\frac{0.05}{4}\right)^{4(5)}$$

$$\approx \$3205.09$$

b)

$$2500\left(1+\frac{0.05}{12}\right)^{12(5)}$$

$$\approx \$3208.4$$

c)

$$A = Pe^{rt}$$

$$= 2500\, e^{0.05(5)}$$

$$\approx \$3210.06$$

$\boxed{5.6}$ Inverse Trigonometric Functions: Differentiation

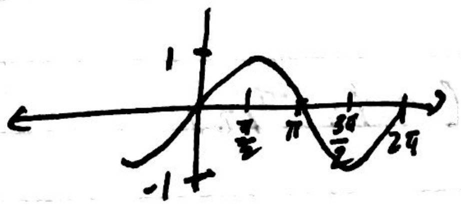
$y = \arcsin x$ if and only if $\sin y = x$

DEFINITIONS OF INVERSE TRIGONOMETRIC FUNCTIONS

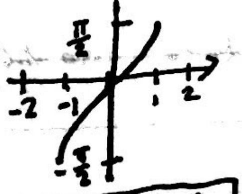
Function			Domain	Range
$y = \sin^{-1}(x)$	iff	$\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \cos^{-1}(x)$	iff	$\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \tan^{-1}(x)$	iff	$\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \cot^{-1}(x)$	iff	$\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \sec^{-1}(x)$	iff	$\sec y = x$	$\lvert x \rvert \ge 1$	$0 \le y \le \pi \quad y \ne \frac{\pi}{2}$
$y = \csc^{-1}(x)$	iff	$\csc y = x$	$\lvert x \rvert \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2} \quad y \ne 0$

graphs of 6 inverse trigonometric functions

e.g. sin function

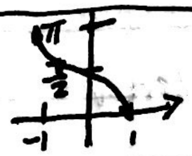


inverse sin function



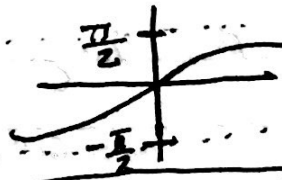
$\boxed{y = \arcsin x}$

Domain: $[-1, 1]$
Range: $[-\pi/2, \pi/2]$



$\boxed{y = \arccos x}$

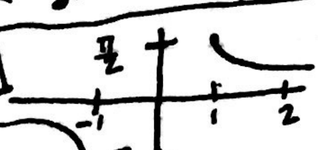
Domain: $[-1, 1]$
Range: $[0, \pi]$



$\boxed{y = \arctan x}$

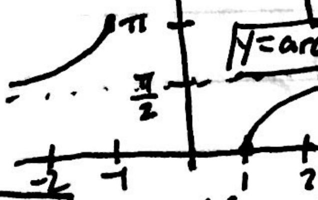
Domain: $(-\infty, \infty)$
Range: $(-\pi/2, \pi/2)$

$\boxed{y = \operatorname{arccsc} x}$



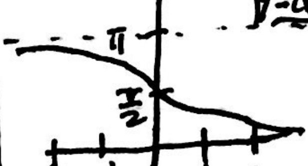
Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[-\pi/2, 0) \cup (0, \pi/2]$

$\boxed{y = \operatorname{arcsec} x}$



Domain: $(-\infty, -1] \cup [1, \infty)$
Rang

$\boxed{y = \operatorname{arccot} x}$



Domain: $(-\infty, \infty)$
Range: $(0, \pi)$

Example 1 EVALUATING INVERSE

a. $\arcsin\left(-\frac{1}{2}\right) = y$

$\sin y = -\frac{1}{2}$

in interval $[-\pi/2, \pi/2]$,

$y = -\frac{\pi}{6}$

$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

b. $\arccos 0 = y$

$0 = \cos y$ $\quad [0, \pi]$

$y = \frac{\pi}{2}$

$\arccos 0 = \frac{\pi}{2}$

c. $\arctan \sqrt{3} = y$

$\sqrt{3} = \tan y$ $\quad (-\pi/2, \pi/2)$

$\arctan \sqrt{3} = \frac{\pi}{3}$

d. Using calculator on radian mode,

$\arcsin(0.3) \approx 0.305$

Inverse Functions have properties

$f(f^{-1}(x))$

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$,

$\sin(\sin^{-1} x) = x$ $\qquad$ $\sin^{-1}(\sin y) = y$

If $-\pi/2 < y < \pi/2$, then

$\tan(\tan^{-1} x) = x$ $\qquad$ $\tan^{-1}(\tan y) = y$

If $|x| \ge 1$; $0 \le y < \pi/2$ or $\pi/2 < y \le \pi$,

$\sec(\sec^{-1} x) = x$ $\qquad$ $\operatorname{arc}\sec(\sec y) = y$

e.g. 2

$\arctan(2x-3) = \frac{\pi}{4}$

$\tan[\arctan(2x-3)] = \tan\frac{\pi}{4}$

$2x-3 = 1$

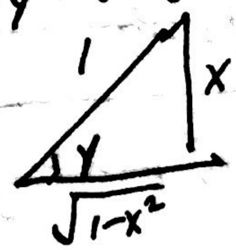
$x = 2$

e.g. 3 USING RIGHT TRIANGLES

a) Given $y = \arcsin x$ $\quad 0 < y < \pi/2$ find $\cos y$

$\sin y = x$

$\cos y = \cos(\arcsin x) = \frac{a}{h} = \sqrt{1-x^2}$



DERIVATIVES OF INVERSE TRIG FUNCTIONS

let u be a differentiable function of x

$\frac{d}{dx}[\sin^{-1} u] = \frac{u'}{\sqrt{1-u^2}}$ $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$

$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$ $\frac{d}{dx}[\text{arccot}\, u] = \frac{-u'}{1+u^2}$

$\frac{d}{dx}[\text{arcsec}\, u] = \frac{u'}{|u|\sqrt{u^2-1}}$ $\frac{d}{dx}[\text{arccsc}\, u] = \frac{-u'}{|u|\sqrt{u^2-1}}$

Eg. 4 Differentiating Inverse Trig Functions

a. $\frac{d}{dx}[\arcsin(2x)] = \frac{2}{\sqrt{1-(2x)^2}} = \frac{2}{\sqrt{1-4x^2}}$

b. $\frac{d}{dx}[\arctan(3x)] = \frac{3}{1+(3x)^2} = \frac{3}{1+9x^2}$

c. $\frac{d}{dx}[\arcsin\sqrt{x}] = \frac{(1/2)x^{-1/2}}{\sqrt{(1-x)}} = \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1}{2\sqrt{x-x^2}}$

d. $\frac{d}{dx}[\text{arcsec}\, e^{2x}] = \frac{2e^{2x}}{e^{2x}\sqrt{(e^{2x})^2-1}} = \frac{2e^{2x}}{e^{2x}\sqrt{e^{4x}-1}} = \frac{2}{\sqrt{e^{4x}-1}}$

e.g. 5 A derivative that can be simplified

$y = \arcsin x + x\sqrt{1-x^2}$

$y' = \frac{1}{\sqrt{1-x^2}} + x\left(\frac{1}{2}\right)(-2x)(1-x^2)^{-1/2} + \sqrt{1-x^2}$

$= \frac{1}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2}$

$= \sqrt{1-x^2} + \sqrt{1-x^2} = \boxed{2\sqrt{1-x^2}}$

e.g. 6 Analyzing an Inverse Trig Graph

$y = (\arctan x)^2$

$y' = 2(\arctan x)\left(\frac{1}{1+x^2}\right)$

$= \frac{2\arctan x}{1+x^2}$

e.g. 7 MAXIMIZING AN ANGLE

Basic differentiation Rules for Elementary Functions

1. $\frac{d}{dx}[cu] = cu'$ 2. $\frac{d}{dx}[u \pm v] = u' \pm v'$ 3. $\frac{d}{dx}[uv] = uv' + vu'$

4. $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$ 5. $\frac{d}{dx}[c] = 0$ 6. $\frac{d}{dx}[u^n] = nu^{n-1}u'$

7. $\frac{d}{dx}[x] = 1$ 8. $\frac{d}{dx}[|u|] = \frac{u}{|u|}(u'), u \neq 0$ 9. $\frac{d}{dx}[\ln u] = \frac{u'}{u}$

10. $\frac{d}{dx}[e^u] = e^u u'$ 11. $\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$ 12. $\frac{d}{dx}[a^u] = (\ln a)a^u u'$

13. $\frac{d}{dx}[\sin u] = (\cos u)u'$ 14. $\frac{d}{dx}[\cos u] = -(\sin u)u'$ 15. $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$

16. $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$ 17. $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$ 18. $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$

19. $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$ 20. $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$ 21. $\frac{d}{dx}[\arctan x] = \frac{u'}{1+u^2}$

22. $\frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$ 23 $\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}}$ 24. $\frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$

3.7 INVERSE TRIGONOMETRIC FUNCTIONS INTEGRATION

Integrals Involving Inverse Trigonometric Functions

1. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + C$
2. $\int \frac{du}{a^2+u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$
3. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a} \text{arcsec} \frac{|u|}{a} + C$

Example ① Integration with Inverse Trigonometric Functions

a. $\int \frac{dx}{\sqrt{4-x^2}} = \arcsin \frac{x}{2} + C$

b. $\int \frac{dx}{2+9x^2} = \frac{1}{3} \int \frac{3\,dx}{(\sqrt{2})^2 + (3x)^2}$

$= \frac{1}{3\sqrt{2}} \arctan \frac{3x}{\sqrt{2}} + C$

c. $\int \frac{dx}{x\sqrt{4x^2-9}} = \int \frac{2\,dx}{2x\sqrt{(2x)^2-3^2}}$

$= \frac{1}{3} \text{arcsec} \frac{|2x|}{3} + C$

Example 2. Integration by substitution

Find $\int \frac{dx}{\sqrt{e^{2x}-1}}$

$u = e^x \Rightarrow du = e^x\,dx \Rightarrow dx = \frac{du}{e^x} = \frac{du}{u}$

$$\int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{dx}{\sqrt{(e^x)^2-1}}$$

$$= \int \frac{du/u}{\sqrt{u^2-1}}$$

$$= \int \frac{du}{u\sqrt{u^2-1}}$$

$$= \operatorname{arcsec} \frac{|u|}{1} + C$$

$$= \operatorname{arcsec} e^x + C$$

e.g. 3 Rewriting as the Sum of 2 quotients

Find $\int \frac{x+2}{\sqrt{4-x^2}}\,dx$

$$\int \frac{x+2}{\sqrt{4-x^2}}\,dx = \int \frac{x}{\sqrt{4-x^2}}\,dx + \int \frac{2}{\sqrt{4-x^2}}\,dx$$

$$= -\frac{1}{2}\int (4-x^2)^{-1/2}(-2x)\,dx + 2\int \frac{1}{\sqrt{4-x^2}}\,dx$$

$$= -\frac{1}{2}\left[\frac{(4-x^2)^{1/2}}{1/2}\right] + 2\arcsin\frac{x}{2} + C$$

$$= -\sqrt{4-x^2} + 2\arcsin\frac{x}{2} + C$$

Completing the Square

$$x^2+bx+c = x^2+bx+\left(\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2+c$$
$$=\left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2+c$$

e.g. 4

Find $\int \frac{dx}{x^2-4x+7}$

$$x^2-4x+7 = (x^2-4x+4)-4+7$$
$$= (x-2)^2+3 = u^2+a^2$$

$$\int \frac{dx}{x^2-4x+7} = \int \frac{dx}{(x-2)^2+3} = \frac{1}{\sqrt{3}} \arctan \frac{x-2}{\sqrt{3}} + C$$

e.g. 5

$f(x) = \frac{1}{\sqrt{3x-x^2}}$ lines $x=\frac{3}{2}$ $x=\frac{9}{4}$

Area $= \int_{3/2}^{9/4} \frac{1}{\sqrt{3x-x^2}}\, dx$

$$\int_{3/2}^{9/4} \frac{dx}{\sqrt{3x-x^2}} = \int_{3/2}^{9/4} \frac{dx}{\sqrt{(3/2)^2-[x-(3/2)]^2}}$$
$$= \arcsin \frac{x-(3/2)}{3/2}\Bigg]_{3/2}^{9/4}$$
$$= \arcsin \frac{1}{2} - \arcsin 0$$
$$= \frac{\pi}{6}$$

BASIC INTEGRATION RULES $(a>0)$

1. $\int k f(u)\,du = k\int f(u)\,du$

2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$

3. $\int du = u + C$

4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C,\ n \neq -1$

5. $\int \frac{du}{u} = \ln|u| + C$

6. $\int e^u\,du = e^u + C$

7. $\int a^u\,du = \left(\frac{1}{\ln a}\right)a^u + C$

8. $\int \sin u\,du = -\cos u + C$

9. $\int \cos u\,du = \sin u + C$

10. $\int \tan u\,du = -\ln|\cos u| + C$

11. $\int \cot u\,du = \ln|\sin u| + C$

12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$

13. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$

14. $\int \sec^2 u\,du = \tan u + C$

15. $\int \csc^2 u\,du = -\cot u + C$

16. $\int \sec u \tan u\,du = \sec u + C$

17. $\int \csc u \cot u\,du = -\csc u + C$

18. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$

19. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$

20. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\,\text{arcsec}\frac{|u|}{a} + C$

Example 6

a. $\int \frac{dx}{x\sqrt{x^2-1}} = \text{arc sec}\,|x| + C$

b. $\int \frac{x\,dx}{\sqrt{x^2-1}} = \frac{1}{2}\int (x^2-1)^{-1/2}(2x)\,dx$

$= \frac{1}{2}\left[\frac{(x^2-1)^{1/2}}{1/2}\right] + C$

$= \sqrt{x^2-1} + C$

Example 7 COMPARING INTEGRATION PROBLEM

a. $\int \frac{dx}{x\ln x} = \int \frac{1/x}{\ln x}\,dx$

$= \ln|\ln x| + C$

b. $\int \frac{\ln x\,dx}{x} = \int \left(\frac{1}{x}\right)(\ln x)\,dx$

$= \frac{(\ln x)^2}{2} + C$